Analytical Results for Dimensionally Regularized Massless On-shell Double Boxes with Arbitrary Indices and Numerators

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Abstract

We present an algorithm for the analytical evaluation of dimensionally regularized massless on-shell double box Feynman diagrams with arbitrary polynomials in numerators and general integer powers of propagators. Recurrence relations following from integration by parts are solved explicitly and any given double box diagram is expressed as a linear combination of two master double boxes and a family of simpler diagrams. The first master double box corresponds to all powers of the propagators equal to one and no numerators, and the second master double box differs from the first one by the second power of the middle propagator. By use of differential relations, the second master double box is expressed through the first one up to a similar linear combination of simpler double boxes so that the analytical evaluation of the first master double box provides explicit analytical results, in terms of polylogarithms $\mathrm{Li}_a(-t/s)$, up to $a = 4$, and generalized polylogarithms $S_{a,b}(-t/s)$, with $a = 1, 2$ and $b = 2$, dependent on the Mandelstam variables s and t, for an arbitrary diagram under consideration.

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1 Introduction

The massless double box diagram shown in Fig. 1 with general numerators and integer powers of propagators is relevant to many important physical processes. The purpose of the present paper is to present an algorithm for the analytical evaluation of the general massless on-shell (i.e. for $p_i^2 = 0,\ i = 1, 2, 3, 4$) double box Feynman diagram in the framework of dimensional regularization [1], with the space-time dimension $d = 4 - 2\epsilon$ as a regularization parameter.

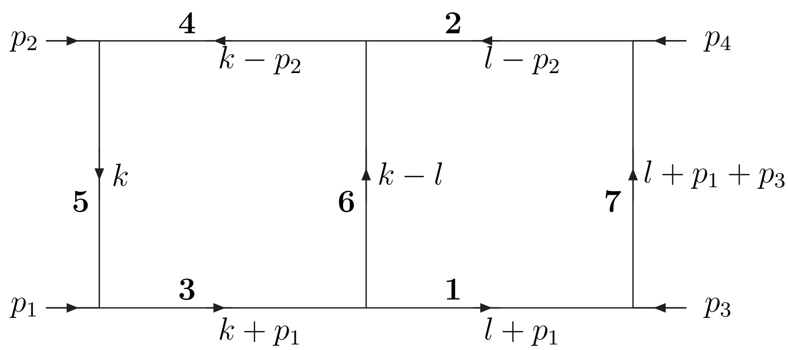

Figure 1: Planar double box diagram.

The dimensionally regularized on-shell master double box Feynman integral (i.e. with all powers of propagators equal to one and no numerators) has been analytically evaluated in [2] in terms of polylogarithms $\mathrm{Li}_a(-t/s)$, up to $a = 4$, and generalized polylogarithms $S_{a,b}(-t/s)$, with $a = 1, 2$ and $b = 2$, dependent on the Mandelstam variables $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$. In [3], recurrence relations within the method of integration by parts (IBP) [4] were explicitly solved and any given double box diagram was expressed as a linear combination of the master double box mentioned above, the second master double box which differs from the first one by the second power of the middle propagator, a family of double boxes with two contracted lines considered in shifted dimension, and vertex diagrams.

In the present paper, we complete this algorithm by evaluating the second master double box and presenting crucial checks of our results by use of asymptotic expansions in the limits $t/s \to 0$ and $s/t \to 0$. In the next section, we present definitions of the double box diagrams through integrals in loop momenta and α-parameters. Then we describe recurrence relations and their solutions in terms of the two master double boxes and a collection of simpler diagrams, in accordance with [3]. In Section 3, we describe how the double boxes with two contracted lines are analytically evaluated in expansion in ϵ up to a desired order. In Section 4, we present the analytical result of ref. [2] for the first master double box and then use differential relations in order to express the second master double box through the first one up to a similar linear combination of simpler double boxes. This provides an explicit analytical result for the second master double box, in terms of the same class of functions as for the first one.

In Section 5, we use the general strategy of regions for expanding the double boxes in the limits $t/s \to 0$ and $s/t \to 0$. We present analytical algorithms for the evaluation of the hard-hard and collinear-collinear contributions to the expansion. We evaluate the LO and NLO (respectively order $1/(s^2 t)$ and $1/(s^3)$) of the expansion of the first double box in the limit $t/s \to 0$ and find an agreement with our explicit result. In conclusion we discuss our results.

2 Recurrence relations and their solution

The general massless on-shell double box Feynman integral in d-dimension can be written as

$$K^{(d)}(P, a_1, \ldots, a_7; s, t) = \int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(\pi^{d/2})^2} \frac{1}{(l^2 + 2p_1 l)^{a_1} (l^2 - 2p_2 l)^{a_2}}$$
$$\times \frac{P(p_1, p_2, p_3, k, l)}{(k^2 + 2p_1 k)^{a_3} (k^2 - 2p_2 k)^{a_4} (k^2)^{a_5} ((k-l)^2)^{a_6} ((l - p_1 - p_3)^2)^{a_7}} , \tag{1}$$

where P is a polynomial, a_i integers, k and l are respectively loop momenta of the left and the right box. Usual prescriptions, $k^2 = k^2 + i0$, $-s = -s - i0$, etc. are implied.

The α-representation of the double box with $P \equiv 1$ is straightforwardly obtained (we omit in the following the polynom P and consider only scalar integrals):

$$K^{(d)}(a_1, \ldots, a_7; s, t) = \frac{(-1)^{a_1 + \ldots + a_7} i^{a_1 + \ldots + a_7 + 2 - d}}{\prod_i \Gamma(a_i)}$$
$$\times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_7 \prod_i \alpha_i^{a_i - 1} D^{-d/2} \exp\left[i \frac{A}{D} s + i \frac{\alpha_5 \alpha_6 \alpha_7}{D} t \right] , \tag{2}$$

where

$$D = (\alpha_1 + \alpha_2 + \alpha_7)(\alpha_3 + \alpha_4 + \alpha_5) + \alpha_6(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7) , \tag{3}$$
$$A = \alpha_1 \alpha_2 (\alpha_3 + \alpha_4 + \alpha_5) + \alpha_3 \alpha_4 (\alpha_1 + \alpha_2 + \alpha_7) + \alpha_6 (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4) . \tag{4}$$

To deal with Feynman integrals with numerators we use the fact that any polynomial in the numerators of the propagators can be represented as a differential operator with respect to some auxiliary parameters (see, e.g., [5]) acting on a scalar diagram. An outcome of this procedure is that any tensor integral is expressed in terms of scalar integrals but in different (shifted) space-time dimensions and with shifted indices of lines (see a detailed discussion in [6]). This step is straightforward and formulae from [6] can be easily programmed on computer.

A more difficult part of the program is to express the so obtained double box integrals (in different dimensions and with all possible sets of indices) in terms of some master integrals and a family of simpler boundary integrals. It turns out that, in our

case, there arise only two master integrals $K^{(d)}_{(1)} = K^{(d)}(1,1,1,1,1,1,1;s,t)$, $K^{(d)}_{(2)} = K^{(d)}(1,1,1,1,1,2,1;s,t)$, and the boundary integrals are either vertex integrals that are evaluated in gamma functions and integrals with at least two reduced lines.

Using the integration by parts method [4] the following relation can be obtained to reduce the index a_1 to unity:

$$sa_1\mathbf{1}^+ = a_7\mathbf{7}^+\mathbf{2}^- + a_6\mathbf{6}^+(\mathbf{2}^- - \mathbf{4}^-) + a_1\mathbf{1}^+\mathbf{2}^- - (d - 2a_2 - a_1 - a_7 - a_6)\,. \quad (5)$$

Hereafter we use the standard notation: $\mathbf{j}^\pm$ is the operator increasing/decreasing the index on the jth line by one unit, i.e. $\mathbf{j}^\pm K(\ldots,a_j,\ldots) = K(\ldots,a_j \pm 1,\ldots)$.

Three similar relations obtained by permutations of lines can be used to reduce indices of lines 1,2,3 and 4 to one. Next we can reduce indices of lines 5 and 7 with the help of the following relations:

$$\begin{aligned}(d-2-2a_5-a_4-a_3)a_5\mathbf{5}^+ &= (d-2-2a_6-a_4-a_3)a_6\mathbf{6}^+ \\ &+(a_5-a_6)a_4\mathbf{4}^+ + (a_5-a_6)a_3\mathbf{3}^+ + a_4a_6\mathbf{4}^+\mathbf{6}^+\mathbf{2}^- + a_3a_6\mathbf{3}^+\mathbf{6}^+\mathbf{1}^-\,, \end{aligned} \quad (6)$$

$$\begin{aligned}(d-2-2a_7-a_2-a_1)a_7\mathbf{7}^+ &= (d-2-2a_6-a_2-a_1)a_6\mathbf{6}^+ \\ &+(a_7-a_6)a_2\mathbf{2}^+ + (a_7-a_6)a_1\mathbf{1}^+ + a_2a_6\mathbf{2}^+\mathbf{6}^+\mathbf{4}^- + a_1a_6\mathbf{1}^+\mathbf{6}^+\mathbf{3}^-. \end{aligned} \quad (7)$$

Using the above recurrence relations we can bring indices of lines 1,2,3,4,5,7 all to unity so that only a_6 can be greater than one.

Our relation to reduce the index of line 6 reads [3]

$$\begin{aligned} & t(d-6-2a_6)(a_6+1)a_6\mathbf{6}^{++} = \\ & \quad -(d-5-a_6)[3d-14-2a_6+2a_6\frac{t}{s}]a_6\mathbf{6}^+ \\ & \quad +\frac{2}{s}(d-4-a_6)^2(d-5-a_6) \\ & \quad +\Big\{(\mathbf{2}^+ + \mathbf{7}^+)\Big[-\frac{2}{s}(d-4-a_6)(d-5-a_6) + 2\frac{t}{s}a_6^2\mathbf{6}^+\Big] \\ & \quad -\Big[2t(a_6+1)a_6\mathbf{6}^{++} + 2(d-4-a_6)a_6\mathbf{6}^+\Big]\mathbf{3}^+\Big\}\mathbf{1}^- \\ & \quad +(d-6)\mathbf{7}^-\mathbf{d}^-\,, \end{aligned} \quad (8)$$

where $\mathbf{d}^-$ decreases the dimension of space-time by 2. Note that the dimension can be effectively shifted only by an even integer number (see detailed discussion in [6]). We stress that this formula is valid only if the indices of lines 1,2,3,4,5,7 were already reduced to unity. Note that in the left-hand side of (8) there is $\mathbf{6}^{++}$, rather than $\mathbf{6}^+$. This means that the index of line 6 cannot be always reduced to one but generally to one or two. One can also get rid of $\mathbf{d}^-$ in the above formulae replacing it by $(\mathbf{1}^+ + \mathbf{2}^+ + \mathbf{6}^+)(\mathbf{3}^+ + \mathbf{4}^+ + \mathbf{5}^+) + \mathbf{6}^+(\mathbf{1}^+ + \mathbf{2}^+)$.

Let us comment shortly on how (8) and other similar relations can be derived. One can start from the integral with numerator $2kp_2$. Since $2kp_2 = (k-p_2)^2 - k^2$ we can eliminate this numerator by canceling lines 4 or 5 (see Fig. 1), and the resulting integrals are simple. On the other hand one can use the machinery mentioned after (4) (see [5, 6]) which expresses tensor integrals as differential operator acting on scalar integral in some dimension $d+2n$. It is more convinient in our case to start from the dimension $d-2$. Then we have

$$(2kp_2)\mathbf{d}^- = -s(\partial_1\partial_6 + \partial_2\partial_3 + \partial_3\partial_6 + \partial_3\partial_7) + t(\partial_6\partial_7), \tag{9}$$

where $\partial_j = \partial/\partial m_j^2$ takes the derivative with respect to the square of the mass on jth line. (After differentiaton all masses are put to zero). In the right-hand side of the above formula, there are scalar integrals with increased indices. Therefore we can apply reduction formulae (5)–(7). The resulting relation will involve terms like $\mathbf{6}^{++}$ and, after some transformation, one can come to (8).

To complete the reduction procedure we should bring the two master integrals (which can appear in shifted dimensions) to the integrals in the generic $d = 4-2\varepsilon$ dimension. Thus we need also a relation that reduces the dimension of the space-time. This can be obtained by inverting the identity

$$K^{(d-2)}(a_1,\ldots) = D(\partial)K^{(d)}(a_1,\ldots), \tag{10}$$

where $K^{(d)}$ is defined in (1), $D(\alpha)$ is given by (3), and ∂ denotes a family of differential operators acting on auxiliary masses of the lines: $\partial_i = \partial/\partial m_i^2$. (After the differentiation, all these masses are put to zero.)

(10) is valid for arbitrary indicies a_i and can be derived from the α-representation of the Feynman integral (see e.g. [6]). This relation however increases the dimension by 2 units. To find a relation decreasing dimension we have to compute operator D^{-1} which is inverse to D. These can be done with the help of already listed above reduction formulae. Instead of giving general form of inverse to (10) it is enough to give it for the master integrals. Since we have two master integrals there are two relations [3]

$$K^{(d)}_{(1)} = \frac{1}{\Delta}\Big[+a_{22}(K^{(d-2)}_{(1)} - f_1^{(d)}K^{(d)}_{(1)}) - a_{12}(K^{(d-2)}_{(2)} - f_2^{(d)}K^{(d)}_{(1)})\Big], \tag{11}$$

$$K^{(d)}_{(2)} = \frac{1}{\Delta}\Big[-a_{21}(K^{(d-2)}_{(1)} - f_1^{(d)}K^{(d)}_{(1)}) + a_{11}(K^{(d-2)}_{(2)} - f_2^{(d)}K^{(d)}_{(1)})\Big], \tag{12}$$

where operators $f_j^{(d)}$ are given by

$$f_1^{(d)} =$$
$$+\Big\{\frac{2}{s}(\mathbf{2}^+\mathbf{3}^+ + \mathbf{2}^+\mathbf{4}^+ + \mathbf{2}^+\mathbf{6}^+ + \mathbf{4}^+\mathbf{6}^+ + \mathbf{4}^+\mathbf{7}^+ + \mathbf{3}^+\mathbf{7}^+)$$

$$
+\frac{4}{s}(\mathbf{2}^+\mathbf{5}^+ + \mathbf{5}^+\mathbf{6}^+ + \mathbf{5}^+\mathbf{7}^+) - \frac{2}{s^2 t}(d-5)(3s+2t)(\mathbf{2}^+ + \mathbf{7}^+)
$$
$$
+\frac{2}{d-6}\mathbf{3}^+\mathbf{6}^+\mathbf{7}^+ - \frac{2}{st(d-6)}\Big(3s(d-5)+t(3d-14)\Big)\mathbf{3}^+\mathbf{6}^+\Big\}\mathbf{1}^-
$$
$$
+\frac{3}{t}\mathbf{7}^-\mathbf{d}^- \,, \tag{13}
$$

$$
f_2^{(d)} =
$$
$$
+\Big\{\frac{2}{s}(\mathbf{2}^+\mathbf{3}^+ + \mathbf{2}^+\mathbf{4}^+ + \mathbf{3}^+\mathbf{7}^+ + \mathbf{4}^+\mathbf{7}^+)\mathbf{6}^+ + \frac{4}{s}(\mathbf{2}^+ + \mathbf{4}^+)\mathbf{6}^{++}
$$
$$
+\frac{2(2d-13)}{s(d-6)}(\mathbf{2}^+ + \mathbf{7}^+ + 2\cdot\mathbf{6}^+)\mathbf{5}^+\mathbf{6}^+
$$
$$
-\frac{2(d-5)(d-7)}{s^2 t(d-6)(d-8)}\Big(s(3d-20)+2t(d-6)\Big)(\mathbf{2}^+ + \mathbf{7}^+)\mathbf{6}^+
$$
$$
+\frac{2(d-5)(d-7)}{s^2 t^2(d-8)}\Big(3s(3d-20)+4t(2d-13)\Big)(\mathbf{2}^+ + \mathbf{7}^+ + \frac{s}{d-6}\mathbf{3}^+\mathbf{6}^+)
$$
$$
+\frac{4}{d-6}(\frac{1}{s}+\mathbf{3}^+)\mathbf{7}^+\mathbf{6}^{++} + \frac{4}{d-8}\Big(\frac{5d-34}{s} + \frac{(3d-20)(2d-13)}{t(d-6)}\Big)\mathbf{3}^+\mathbf{6}^{++}\Big\}\mathbf{1}^-
$$
$$
+\Big\{\frac{3d-20}{t(d-6)}\mathbf{6}^+ - \frac{d-7}{st^2(d-8)}\Big(3s(3d-20)+4t(2d-13)\Big)\Big\}\mathbf{7}^-\mathbf{d}^- \,, \tag{14}
$$

$$
a_{11} = \frac{2}{s^2 t}(d-5)^2(3s+2t), \tag{15}
$$
$$
a_{12} = -\frac{2}{s}(4d-21) - \frac{3}{t}(3d-16), \tag{16}
$$
$$
a_{21} = -\frac{(d-5)^2(d-7)}{st(d-8)}\Big(\frac{8(2d-13)}{s} + \frac{6(3d-20)}{t}\Big), \tag{17}
$$
$$
a_{22} = \frac{d-7}{s^2 t^2(d-8)}\Big(3s^2(3d-16)(3d-20) + 6st(5d^2-59d+172)
$$
$$
+4t^2(d-5)(d-6)\Big), \tag{18}
$$
$$
\Delta = \frac{16(s+t)(d-5)^3(d-6)(d-7)}{s^4 t(d-8)} \,. \tag{19}
$$

Formulae (5)–(19) solve the problem of the reduction of any planar double box to two master integrals $K_{(1)}$, $K_{(2)}$ and a set of simpler integrals with reduced lines. From now on we omit the superscript (d) because we shall deal with integrals in $d = 4 - 2\epsilon$ (non-shifted) dimensions.

We have checked the reduction scheme (5)–(19) by expanding the integrand in two regimes, $s/t \to 0$ and $t/s \to 0$, and evaluating the resulting one-scale integrals by the method described in Sect. 5.

3 Double boxes with two contracted lines

In the framework of the reduction procedure presented in the previous section, the boundary values for general double boxes are either two master double boxes, or vertex diagrams, i.e. at $a_5 = 0$ or $a_7 = 0$, or double boxes with two contracted lines. The latter can be of the following two types: $a_1 = a_4 = 0$ (or the symmetrical variant $a_2 = a_3 = 0$) shown in Fig.2, or $a_3 = a_4 = 0$ (or the symmetrical variant $a_1 = a_2 = 0$) shown in Fig.3. Let us call them respectively the box with a diagonal and the box with a one-loop insertion. Note that they generally arise in a shifted dimension.

We should mention that there are no integrals with only one contracted line left. As far as one of the lines 1,2,3 or 4 is contracted one can still proceed with reduction formulae from Sect. 2 or apply the standard "rule of triangle" [4] to reduce one more line. Thus the two master double boxes plus boxes with two contracted line form the basis of integrals.

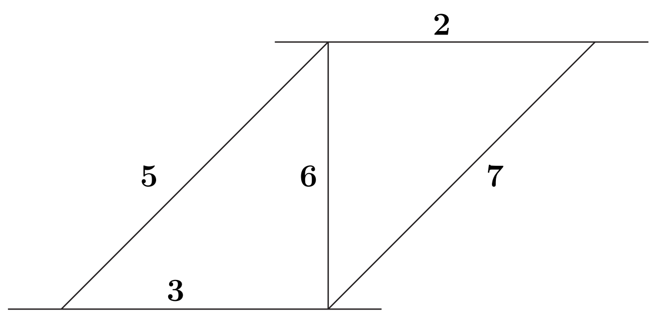

Figure 2: A box with a diagonal.

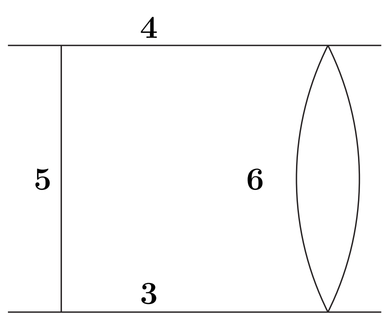

Figure 3: A box with a one-loop insertion.

These cases without two lines are much simpler than the master double boxes. Using α-representation (2) and representing one of the functions involved into the

Mellin–Barnes (MB) integral

$$\frac{1}{(X+Y)^\nu} = \frac{1}{\Gamma(\nu)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}w \frac{Y^w}{X^{\nu+w}}\Gamma(\nu+w)\Gamma(-w)\;, \tag{20}$$

we obtain the following results:

$$\begin{aligned} K^{(d)}(0,a_2,a_3,0,a_5,a_6,a_7;d+n;s,t) \;&=\; \frac{i^2(-1)^a}{\prod\Gamma(a_i)(-s)^{a-n-4+2\epsilon}} \\ &\times\frac{\Gamma(2-a_3-a_5-\epsilon+n/2)\Gamma(2-a_6-\epsilon+n/2)\Gamma(2-a_2-a_7-\epsilon+n/2)}{\Gamma(6-a+3n/2-3\epsilon)\Gamma(4-a_3-a_5-a_6+n-2\epsilon)\Gamma(4-a_2-a_6-a_7+n-2\epsilon)} \\ &\times\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\mathrm{d}z\,(t/s)^z\Gamma(a_5+z)\Gamma(a_7+z)\Gamma(a-4-n+2\epsilon+z)\Gamma(-z) \\ &\times\Gamma(4-a_2-a_5-a_6-a_7+n-2\epsilon-z)\Gamma(4-a_3-a_5-a_6-a_7+n-2\epsilon-z)\;, \end{aligned} \tag{21}$$

and

$$\begin{aligned} K(a_1,a_2,0,0,a_5,a_6,a_7;d+n;s,t) \;&=\; \frac{i^2(-1)^a}{\prod\Gamma(a_i)(-s)^{a-n-4+2\epsilon}} \\ &\times\frac{\Gamma(2-a_5-\epsilon+n/2)\Gamma(2-a_6-\epsilon+n/2)}{\Gamma(6-a+3n/2-3\epsilon)\Gamma(4-a_5-a_6+n-2\epsilon)} \\ &\times\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\mathrm{d}z\,(t/s)^z\Gamma(a_7+z)\Gamma(-4+a-n+2\epsilon+z) \\ &\times\Gamma(-2+a_5+a_6+\epsilon-n/2+z)\Gamma(4-a_1-a_5-a_6-a_7+n-2\epsilon-z) \\ &\times\Gamma(4-a_2-a_5-a_6-a_7+n-2\epsilon-z)\Gamma(-z)\,, \end{aligned} \tag{22}$$

where $a=\sum_i a_i$. The contour of integration is chosen in the standard way: the poles with the $\Gamma(\ldots+z)$-dependence are to the left of the contour and the poles with the $\Gamma(\ldots-z)$-dependence are to the right of it.

Then each of these boundary integrals with given values of integer indices is decomposed into 'singular' and 'regular' parts. For the diagonal crossed boxes, the singular part is written as minus the sum of the residua of the integrand at the points $j-2\epsilon$, with $j=-\max\{a_2,a_3\}-a_5-a_6-a_7+4+n,\ldots,-1$, plus the sum of the residua of the integrand at the points $j-2\epsilon$ for $j=0,\ldots,4+n-a_2-a_3-a_5-a_6-a_7$. For the boxes with one-loop insertions, the singular part is written as minus the sum of the residua of the integrand at the points $j-2\epsilon$, with $j=-\max\{a_1,a_2\}-a_5-a_6-a_7+4+n,\ldots,-1$, plus the sum of the residua of the integrand at the points $j-2\epsilon$ for $j=0,\ldots,4+n-a_1-a_2-a_5-a_6-a_7$.

The regular parts are given by MB integrals where no gluing of poles of gamma functions with $+z$ and $-z$ dependence arises. They can be written as MB integrals for $-1<\mathrm{Re}(z)<0$ with an integrand expanded in a Laurent series in ϵ up to a desired order. Then these integrals are straightforwardly evaluated by closing the contour of integration to the right and taking residua at the points $z=0,1,2,\ldots$. At this step,

one can use a collection of formulae for summing up series presented in ref. [9]. The evaluation of both the singular and the regular parts is easily realized on computer.

Note that in the recurrence relations of the previous sections, these boundary double boxes can arise with coefficients that involve poles up to the second order in ϵ so that the expansion up to ϵ^2 is here necessary. However the 'master' boxes with a diagonal or one-loop insertion enter with coefficients finite in ϵ so that it is sufficient to compute then, in expansion in ϵ, up to the finite part. Let us, for example, present an analytical result for the master box with a diagonal in d dimensions:

$$K(0,1,1,0,1,1,1;d;s,t) = \frac{(i\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon})^2}{-s-t} K_0(s,t,\epsilon)\,, \tag{23}$$

where

$$\begin{aligned}
K_0(s,t,\epsilon) \;\; = \;\; & -\left(\ln^2(t/s)+\pi^2\right)\frac{1}{2\epsilon^2} \\
&+\Big[2\mathrm{Li}_3\left(-t/s\right) - 2\ln(t/s)\mathrm{Li}_2\left(-t/s\right) - \left(\ln^2(t/s)+\pi^2\right)\ln(1+t/s) \\
&+\frac{2}{3}\ln^3(t/s) + \ln(-s)\ln^2(t/s) + \pi^2\ln(-t) - 2\zeta(3)\Big]\frac{1}{\epsilon} \\
&+4\left(S_{2,2}(-t/s) - \ln(t/s)S_{1,2}(-t/s)\right) - 4\mathrm{Li}_4\left(-t/s\right) \\
&+4\left(\ln(1+t/s) - \ln(-s)\right)\mathrm{Li}_3\left(-t/s\right) \\
&+2\left(\ln^2(t/s) + 2\ln(-s)\ln(t/s) - 2\ln(t/s)\ln(1+t/s)\right)\mathrm{Li}_2\left(-t/s\right) \\
&+2\left(\frac{2}{3}\ln^3(t/s) + \ln(-s)\ln^2(t/s) + \pi^2\ln(-t) - 2\zeta(3)\right)\ln(1+t/s) \\
&-\left(\ln^2(t/s)+\pi^2\right)\ln^2(1+t/s) - \frac{1}{2}\ln^4(t/s) - \frac{4}{3}\ln(-s)\ln^3(t/s) \\
&-\left(\ln^2(-s) + \frac{11}{12}\pi^2\right)\ln^2(t/s) - \pi^2\ln^2(-s) - 2\pi^2\ln(-s)\ln(t/s) \\
&+4\zeta(3)\ln(-t) - \frac{\pi^4}{20}\,.
\end{aligned} \tag{24}$$

Here $\mathrm{Li}_a\left(z\right)$ is the polylogarithm [7] and

$$S_{a,b}(z) = \frac{(-1)^{a+b-1}}{(a-1)!b!}\int_0^1 \frac{\ln^{a-1}(t)\ln^b(1-zt)}{t}\mathrm{d}t \tag{25}$$

a generalized polylogarithm [8]. Using known formulae that relate polylogarithms and generalized polylogarithms with arguments z and $1/z$ [7, 8] one can rewrite this and similar results for the master double boxes in terms of the same class of functions depending on the inverse ratio s/t.

We do not give explicit result for the box with a 1-loop insertion since it is certainly a simpler object. Indeed the 1-loop onsertion can be trivially integrated and we have a 1-loop box where one of the indicies is equal to ε. Such kind of integrals can be expressed using standard methods in terms of the hypergeometric function ${}_2F_1$ (see e.g. [10]).

4 Master double boxes

The first master double box

$$K(1,\ldots,1;d,s,t) = \frac{\left(i\mathrm{e}^{-\gamma_{\rm E}\epsilon}\right)^2}{(-s)^{2+2\epsilon}(-t)} K_1(t/s,\epsilon)\,, \tag{26}$$

has been evaluated in ref. [2] by use of α-parameters and resolving singularities in a 5-fold MB integral:

$$\begin{aligned} K_1(x,\epsilon) \;\; &= \;\; -\frac{4}{\epsilon^4} + \frac{5\ln x}{\epsilon^3} - \left(2\ln^2 x - \frac{5}{2}\pi^2\right)\frac{1}{\epsilon^2} \\ &- \left(\frac{2}{3}\ln^3 x + \frac{11}{2}\pi^2 \ln x - \frac{65}{3}\zeta(3)\right)\frac{1}{\epsilon} + \frac{4}{3}\ln^4 x + 6\pi^2 \ln^2 x - \frac{88}{3}\zeta(3)\ln x + \frac{29}{30}\pi^4 \\ &- \left[2\mathrm{Li}_3(-x) - 2\ln x \mathrm{Li}_2(-x) - \left(\ln^2 x + \pi^2\right)\ln(1+x)\right]\frac{2}{\epsilon} \\ &-4\left(S_{2,2}(-x) - \ln x S_{1,2}(-x)\right) + 44\mathrm{Li}_4(-x) - 4\left(\ln(1+x) + 6\ln x\right)\mathrm{Li}_3(-x) \\ &+2\left(\ln^2 x + 2\ln x\ln(1+x) + \frac{10}{3}\pi^2\right)\mathrm{Li}_2(-x) \\ &+\left(\ln^2 x + \pi^2\right)\ln^2(1+x) - \frac{2}{3}\left(4\ln^3 x + 5\pi^2 \ln x - 6\zeta(3)\right)\ln(1+x)\,. \end{aligned} \tag{27}$$

To evaluate the second master double box, i.e.

$$K(1,1,1,1,1,2,1;d;s,t) = \frac{\left(i\mathrm{e}^{-\gamma_{\rm E}\epsilon}\right)^2}{(-s)^{2+2\epsilon}t^2} K_2(t/s,\epsilon)\,, \tag{28}$$

let us take first derivatives in t of the two master double boxes. Using α-representation (2) we obtain

$$\frac{\partial}{\partial t}K(1,\ldots,1;d;s,t) \;\; = \;\; -K(1,1,1,1,2,2,2;d+2;s,t)\,, \tag{29}$$

$$\frac{\partial}{\partial t}K(1,1,1,1,1,2,1;d;s,t) \;\; = \;\; -2K(1,1,1,1,2,3,2;d+2;s,t)\,. \tag{30}$$

We now use the results presented in Sect. 2 to express both right-hand sides as linear combinations of the two master double boxes, vertex diagrams (two in the first case and three in the second case) and a numerous family (around fifty terms in each case) of diagonal crossed boxes and boxes with one-loop insertions. Substituting explicit result (27) into the first equation and evaluating all the terms in the right hand side as explained in Section 3 we obtain an analytical result for the second master double box:

$$\begin{aligned} K_2(x,\epsilon) \;\; &= \;\; \frac{4}{\epsilon^4} - 5\left(\ln x - 2\right)\frac{1}{\epsilon^3} + \left(2\ln^2 x - 14\ln x - \frac{5}{2}(\pi^2+4)\right)\frac{1}{\epsilon^2} \\ &+\left(\frac{2}{3}\ln^3 x + 8\ln^2 x + \left(\frac{11}{2}\pi^2 + 14\right)\ln x - 2 - 3\pi^2 - \frac{65}{3}\zeta(3)\right)\frac{1}{\epsilon} \end{aligned}$$

$$
\begin{aligned}
&-\frac{4}{3}\ln^3 x(\ln x+1)-2\left(3\pi^2+4\right)\ln^2 x+\left(10+9\pi^2+\frac{88}{3}\zeta(3)\right)\ln x\\
&+20+12\pi^2-\frac{29}{30}\pi^4+\frac{4}{3}\zeta(3)\\
&+x\left[-\frac{7}{\epsilon^3}+(8\ln x-33)\frac{1}{\epsilon^2}+\left(26\ln x+6+\frac{21}{2}\pi^2\right)\frac{1}{\epsilon}\right.\\
&\left.+\frac{1}{6}\left(-32\ln^3 x-4(21+26\pi^2)\ln x+180+209\pi^2+904\zeta(3)\right)\right]\\
&+\left[2\mathrm{Li}_3\left(-x\right)-2\ln x\mathrm{Li}_2\left(-x\right)-\left(\ln^2 x+\pi^2\right)\ln(1+x)\right]\frac{2}{\epsilon}\\
&-4x\left[8\left(\mathrm{Li}_3\left(-x\right)-\ln x\mathrm{Li}_2\left(-x\right)\right)-4\left(\ln^2 x+\pi^2\right)\ln(1+x)\right]\\
&+4\left(S_{2,2}(-x)-\ln xS_{1,2}(-x)\right)-44\mathrm{Li}_4\left(-x\right)+4\left(\ln(1+x)+6\ln x-2\right)\mathrm{Li}_3\left(-x\right)\\
&-2\left(\ln^2 x+2\ln x\ln(1+x)-4\ln x+\frac{10}{3}\pi^2\right)\mathrm{Li}_2\left(-x\right)-\left(\ln^2 x+\pi^2\right)\ln^2(1+x)\\
&+\left(\frac{8}{3}\ln^3 x+4\ln^2 x+\frac{10}{3}\pi^2\ln x+4\pi^2-4\zeta(3)\right)\ln(1+x)\,. \qquad (31)
\end{aligned}
$$

Proceeding in the same way with the second recurrence relation (30), and inserting there our analytical results for the master double boxes we eventually obtain an identity of the left-hand and the right-hand sides. This fact turns out to be a very non-trivial check of the recurrence relations, their solutions and our analytical expressions for the master double boxes.

5 Asymptotic expansions of the double box

We still want other checks and are going to compare our results with what can be obtained by expanding the first master double box in various limits. To expand the double box diagrams in the limit $t \to 0$ let us use the strategy of regions:

(*i*) Consider various regions of the loop momenta and expand, in every region, the integrand in a Taylor series with respect to the parameters that are considered small in the given region;

(*ii*) Integrate the integrand expanded, in every region in its own way, over the whole integration domain in the loop momenta;

(*iii*) Put to zero any scaleless integral.

In the off-shell and off-threshold limits, this strategy leads to the well-known explicit prescriptions [11] (see a brief review [12]) based on the strategy of subgraphs. Although the strategy of subgraphs was successfully applied to some on-shell limits [13, 14], the strategy of regions looks generally more flexible. In particular, it proved to be adequate for constructing the threshold expansion [15].

Let us choose, for convenience, the external momenta as follows:

$$
p_{1,2}=(\mp Q/2,Q/2,0,0),\; r\equiv p_1+p_3=(T/Q,0,\sqrt{T+T^2/Q^2},0)\,, \qquad (32)
$$

where $s=-Q^2$ and $t=-T$. The given limit is closely related to the Sudakov limit so that the following standard regions happen to be typical for it:

$$\begin{aligned}
\textit{hard}\ \text{(h)}: &\quad k \sim Q\,, \\
\textit{1-collinear}\ \text{(1c)}: &\quad k_+ \sim Q,\ k_- \sim T/Q\,,\ \underline{k} \sim \sqrt{T}\,, \\
\textit{2-collinear}\ \text{(2c)}: &\quad k_- \sim Q,\ k_+ \sim T/Q\,,\ \underline{k} \sim \sqrt{T}\,, \\
\textit{ultrasoft}\ \text{(us)}: &\quad k \sim T\,.
\end{aligned}$$

Here $k_\pm = k_0 \pm k_1$, $\underline{k} = (k_2, k_3)$. We mean by $k \sim Q$, etc. that any component of k_μ is of order Q.

It turns out that the (h-h), (1c-1c) and (2c-2c) are the only non-zero contributions to the asymptotic expansion in the limit $t/s \to 0$. In particular, all the (c-h) contributions and all the contributions with ultrasoft momenta are zero because they generate scaleless integrals.

The (h-h) region generates the contribution given by Taylor expansion of the integrand in the vector r. Every diagram from this contribution corresponds to the forward scattering configuration, $p_3 = -p_1$ and $p_4 = -p_2$, and can be evaluated for general ϵ in gamma functions by resolving recurrence relations following from integration by parts [4]. The first step of this procedure is to reduce an index a_5 or a_7 to zero and thereby obtain vertex massless diagrams. The latter reduction, in the scalar case, was constructed in ref. [14]. (In the case without numerators, the reduction of the forward scattering double boxes was presented in ref. [16].)

We have constructed two different procedures for the evaluation of the (h-h) part: along the lines of this standard recursion and also by expanding the integrand of the α-representation in the variable t and using tricks with shifting dimension. We have implemented both methods and checked that they give the same results for first several coefficients.

We describe now how one can expand the integrals in the variable t. The most suitable method to achieve this is the one proposed in [17]. For the expansion in t/s, we have

$$K^{(d)}(s,t) \stackrel{\text{h--h}}{=} \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{t}{s}\right)^j Q_t^j K^{(d+2j)}(s,0)\,, \tag{33}$$

where Q_t is the differential operator acting on the masses of the lines. If we denote $\partial/\partial m_i^2$ as ∂_i then

$$Q_t = \partial_5 \partial_6 \partial_7\,. \tag{34}$$

After the differentiation in (33), all these auxiliary masses are put to zero. As a result each coefficient in the expansion (33) consists of integrals depending only on one scale s. These can already be evaluated analytically by the standard "triangle" rule, i.e. using the integration by parts [4].

The integrals $K^{(d+2j)}(s,0)$ belong to the class of primitives, i.e. they can be evaluated in terms of Γ-functions. The problem here is that the repeated application of the triangle rule brings more and more powers of $1/\varepsilon$ and therefore a deeper expansion of Γ-functions is required. To keep things under control one can use e.g. an algorithm described below. With the help of this algorithm, the depth of the ε-expansion is kept at the level of 6. The reduction proceeds as follows:

(i) Use the relation

$$\begin{aligned}&(2d-2a_5-2a_6-2a_7-a_1-a_2-a_3-a_4)\\&=a_1\mathbf{1}^+\mathbf{7}^-+a_2\mathbf{2}^+\mathbf{7}^-+a_3\mathbf{3}^+\mathbf{5}^-+a_4\mathbf{4}^+\mathbf{5}^-\,.\end{aligned}\tag{35}$$

(This is the same first step as in the above mentioned standard recursive procedure.) With its help, we can get rid of either line 5 or 7 and thereby reduce the double box to a planar vertex. Relation (35) has the feature that the left-hand side passes only once through the "critical point" (when the expression in parentheses is proportional to ε). Therefore at most a single power of $1/\varepsilon$ will be generated in the course of this step of the recursion.

(ii) Reduce indices of the lines 3 and 4 to one with the help of

$$sa_3\mathbf{3}^+=a_5\mathbf{5}^+\mathbf{4}^-+a_6\mathbf{6}^+(\mathbf{4}^--\mathbf{2}^-)+a_3\mathbf{3}^+\mathbf{4}^--(d-2a_4-a_3-a_5-a_6)\,,\tag{36}$$
$$sa_4\mathbf{4}^+=a_5\mathbf{5}^+\mathbf{3}^-+a_6\mathbf{6}^+(\mathbf{3}^--\mathbf{1}^-)+a_4\mathbf{4}^+\mathbf{3}^--(d-2a_3-a_4-a_5-a_6)\,,\tag{37}$$

This step brings no new powers of $1/\varepsilon$.

(iii) Shrink the line 4 (or 3) using the triangle rule

$$(d-2a_6-a_1-a_2)=a_1\mathbf{1}^+(\mathbf{6}^--\mathbf{3}^-)+a_2\mathbf{2}^+(\mathbf{6}^--\mathbf{4}^-)\,.\tag{38}$$

Here at most one additional power of $1/\varepsilon$ is generated.

(iv) The index of the line 1 is then reduced to unity by

$$a_1\mathbf{1}^+=(d-1-2a_7-a_6-a_1-a_2)\mathbf{7}^++a_6\mathbf{6}^+\mathbf{7}^+\mathbf{5}^--a_2\mathbf{2}^+-a_6\mathbf{6}^+\,.\tag{39}$$

No new powers of $1/\varepsilon$ arise here. Note that the line 7 could be absent before this step but now it appears again due to the $\mathbf{7}^+$ term.

(v) Apply the triangle rule

$$(d-2a_3-a_5-a_6)=a_6\mathbf{6}^+(\mathbf{3}^--\mathbf{1}^-)+a_5\mathbf{5}^+\mathbf{3}^-\tag{40}$$

to shrink either line 1 or 3. Note that a dangerous term $\mathbf{6}^+\mathbf{1}^-$ which leads to the oscillation of the left-hand side around ε is harmless here because the index of the line 1 is already equal to unity due to (iv).

(vi) Now we have to evaluate the diagram without lines 1 and 4. (Other cases are trivial). To simplify it we use (35) which now looks like

$$(2d-2a_5-2a_6-2a_7-a_2-a_3)=a_2\mathbf{2}^+\mathbf{7}^-+a_3\mathbf{3}^+\mathbf{5}^-\,.\tag{41}$$

This completes the algorithm.

Thus steps (i),(iii) and (vi) in the worst case bring a factor $1/\epsilon$ each. Other three powers of $1/\epsilon$ arise when evaluating primitive integrals. Therefore the total depth of the ε-expansion of the Γ-functions must be not greater than six. Note that diagrams without line 6 must be separated from the very beginning and evaluated immediately without any recursion in terms of Γ-functions. The point is that when evaluating primitive diagrams with $a_6 = 0$ we obtain poles of the fourth order instead of usual third order poles.

Let us now describe how an arbitrary term of the (c-c) contribution to the expansion of the (first) master double box can be evaluated. If we consider k and l 1-collinear then k^2, l^2, p_2k and p_2l are of order T while p_1k and p_1l are of order Q^2. Moreover, $(l+r)^2 \equiv l^2 + 2lr - T \sim (l+\tilde{r})^2$, where $r = p_1 + p_3$ and

$$\tilde{r} = (T/(2Q), -T/(2Q), \sqrt{T}, 0)\,, \tag{42}$$

with $2p_1\tilde{r} = 0$, $2p_2\tilde{r} = \tilde{r}^2 = -T$. Thus the (1c-1c) contribution is obtained by expanding propagators $1/(k^2+2p_1k)$ and $1/(l^2+2p_1l)$ in Taylor series respectively in k^2 and l^2, and by expansion also in Taylor series in $2p_1r$. (Note that we are dealing with a function of three kinematical variables, $2p_1r$, $2p_2r$ and r^2. So, we expand the integrand (e.g. in the α-representation) in $2p_1r$ and then put $2p_1r = T$.) Actually we want to expand the master box with all a_i equal to 1. Therefore only the leading term in the Taylor expansion in k^2 is non-zero because, starting from the next order, the factor k^2 cancels the propagator $1/k^2$ and we obtain a zero scaleless integral.

In the following it is more convenient to consider integrals with general arbitrary indicies a_i. Thus, we need integrals of the type

$$\begin{aligned} J(a_1,\ldots,a_9;d,s,t) &= \frac{1}{a_8!}\left(\frac{\partial}{\partial X}\right)^{a_8} \int\int \frac{\mathrm{d}^d k \mathrm{d}^d l}{(\pi^{d/2})^2(-2p_1l)^{a_1}(-l^2+2p_2l)^{a_2}} \\ &\times \left.\frac{(l^2)^{a_9}}{(-2p_1k)^{a_3}(-k^2+2p_2k)^{a_4}(-k^2)^{a_5}(-(k-l)^2)^{a_6}(-(l-r)^2)^{a_7}}\right|_{X=0}\,, \end{aligned} \tag{43}$$

with $X = 2p_1r$, for integer a_i. However this integral taken alone is not generally regularized dimensionally. Only if we add the corresponding symmetrical contribution (i.e. for $a_1 \leftrightarrow a_2$, $a_3 \leftrightarrow a_4$) we shall have a result that exists within dimensional regularization. An efficient way to deal with this problem is to introduce an auxiliary analytical regularization which allows to consider the above terms separately. Let us introduce it into the lines 1 and 2 (although we could choose as well 3 and 4, or all four of these lines), i.e. with $a_1 \to a_1 + x_1, a_2 \to a_2 + x_2$ plus a symmetrical contribution which is given by interchanging $a_1 + x_1$ and $a_2 + x_2$. Only in the sum we may switch off this regularization, i.e. let $x_1 \to 0, x_2 \to 0$.

The general integral (43) at $a_8 = a_9 = 0$ can be represented as

$$J(a_1,\ldots,a_7,0,0;d,s,t) = \frac{i^2(-1)^a\Gamma(a'-d)}{\prod_{i\neq 1,3}\Gamma(a_i)(Q^2)^{a_1+a_3}T^{a'-d}}$$

$$\times \int \mathrm{d}\underline{\alpha}'\delta\left(\sum_{i\neq 1,3}\alpha_i - 1\right)\prod_{i\neq 1,3}\alpha_i^{a_i-1}(\alpha_5\alpha_6\alpha_7)^{d-a'}\bar{D}^{a-\frac{3}{2}d}D_1^{-a_1}D_3^{-a_3}\,, \quad (44)$$

where

$$\begin{aligned}
\bar{D} &= (\alpha_2+\alpha_7)(\alpha_4+\alpha_5)+\alpha_6(\alpha_2+\alpha_4+\alpha_5+\alpha_7)\,, && (45)\\
D_1 &= \alpha_2(\alpha_4+\alpha_5)+\alpha_6(\alpha_2+\alpha_4)\,, && (46)\\
D_3 &= \alpha_4(\alpha_2+\alpha_7)+\alpha_6(\alpha_2+\alpha_4)\,. && (47)\\
& && (48)
\end{aligned}$$

Moreover, the integral above is represented in terms of parameters $\underline{\alpha}' = \{\alpha_i\,, i = 2,4,5,6,7\}\,, a = \sum a_i\,,$ $a' = \sum_{i\neq 1,3} a_i\,.$ In the argument of the delta function one can put the sum of an arbitrary subset of $\alpha_i\,, i = 2,4,5,6,7$.

The integrals with $a_8 > 0$ are obtained by the replacement

$$(Q^2)^{-a_1-a_3}D_1^{-a_1}D_3^{-a_3} \to (Q^2 D_1 + XB_1)^{-a_1}(Q^2 D_3 + XB_3)^{-a_3}\,,$$

where $B_1 = \alpha_7(\alpha_4+\alpha_5+\alpha_6)$ and $B_3 = \alpha_6\alpha_7$, subsequent expansion in Taylor series in X and keeping terms of order X^{a_8}.

Starting from (44) and it generalization for $a_8 > 0$ we can arrive at a triple MB integral in the following way:

1. Choose the delta function as $\delta(\alpha_4+\alpha_6-1)$. 2. Represent $1/(D_3)^{a_3+\cdots}$ in a MB representation, using the decomposition $D_3 = D_0 + \alpha_4\alpha_7$. 3. Integrate in α_7. Note that the function $D_1+\alpha_5\alpha_6$ arises. 4. Represent $(D_1+\alpha_5\alpha_6)^{\cdots}$ via MB representation. 5. Represent $(D_1 = D_0+\alpha_2\alpha_5)^{\cdots}$ via MB representation. 6. Integrate in α_5 and then in α_2. 7. Integrate in $\xi = \alpha_4$ with $\alpha_6 = 1-\xi$.

Then we obtain a triple MB integral of a ratio of Γ-functions. This integral is evaluated, in expansion in ϵ, by the standard technique of shifting contours and expansion in MB integrals. First, singularities in $x_1 - x_2$ are localized. Second, the same technique is applied for picking up singularities in ϵ. As a result we end up with a collection of explicit terms plus integrals which are finite in ε. These last integrals can be expanded in ε up to the desired order. So, in the end, various integrals with Γ, ψ and their derivatives are evaluated (see examples in [2]).

The integrals with $a_9 > 0$ can be represented in α-parameters in a cumbersome way. Using IBP [4] (starting from the integral of $(\partial/\partial k)\ k$) we obtain the following recurrence relation:

$$\left(d - a_3 - a_4 - 2a_5 - a_6 - a_4\mathbf{4}^+\mathbf{5}^- - a_6\mathbf{6}^+(\mathbf{5}^- - \mathbf{9}^+)\right) J = 0\,. \quad (49)$$

Note that our general integral is zero when at least one of parameters a_5, a_6, a_7 is a non-positive integer so that this relation is of no use in its primary form. However we may turn to a new integral which is defined without $\Gamma(a_6)$ in the denominator, i.e. put

$J(\ldots a6,\ldots) = J'(\ldots a6,\ldots)/\Gamma(a_6)$. Then (49) applied to the new integrals J' can be used to decrease the number a_9 to zero. We shall generally deal with non-positive a_6 after that.

To evaluate the (c-c) contribution to the expansion of the first master double box it suffices to consider $a_2 = a_3 = a_4 = a_5 = a_7 = 1$, and general integer a_1, a_6 and $a_8, a_9 \geq 0$. Then a_6 can be reduced to zero using (49). In particular, in the leading order of the expansion we meet $J(1,\ldots,1,0,0)$ and, in the NLO, $J(2,1\ldots,1,0,1)$ and $J(1,1\ldots,1,1,0)$. Using (49) we then express $J(2,1\ldots,1,0,1)$ through $J(2,1,1,1,1,0,1,0,0)$ (defined at negative a_6 as explained above).

We have evaluated the LO and NLO of the (c-c) contribution (and added the LO order of the (h-h) contribution which is really NLO for the whole expansion) of the master double box integral in the limit $t/s \to 0$. After that we have found complete agreement with the first two terms of the expansion of the explicit result (27):

$$\begin{aligned}
K_1(x,\epsilon) \;=\; & -\frac{4}{\epsilon^4} + \frac{5\ln x}{\epsilon^3} - \left(2\ln^2 x - \frac{5}{2}\pi^2\right)\frac{1}{\epsilon^2} \\
& - \left(\frac{2}{3}\ln^3 x + \frac{11}{2}\pi^2 \ln x - \frac{65}{3}\zeta(3)\right)\frac{1}{\epsilon} \\
& + \frac{4}{3}\ln^4 x + 6\pi^2 \ln^2 x - \frac{88}{3}\zeta(3)\ln x + \frac{29}{30}\pi^4 \\
& + 2x\left[\frac{1}{\epsilon}\left(\ln^2 x - 2\ln x + \pi^2 + 2\right)\right. \\
& \left. - \frac{1}{3}\left(4\ln^3 x + 3\ln^2 x + (5\pi^2 - 36)\ln x + 2(33 + 5\pi^2 - 3\zeta(3))\right)\right] \\
& + O(x^2 \ln^3 x)\,. \qquad (50)
\end{aligned}$$

The expansion in the limit $s/t \to 0$ has the same structure as in the previous case: only (h-h) and (c-c) contributions are non-zero. However, in this case, there are three (c-c) contributions, and the poles with respect to an auxiliary parameter of analytic regularization happen to be up to the second order (as in the case of the non-planar vertex diagram in the Sudakov limit — see [18]).

6 Conclusion

The on-shell double box provides a curious example where the analytical evaluation is simpler than the evaluation by means of asymptotic expansions in some limits (in this case, $t/s \to 0$ and $s/t \to 0$). We have met rather inconvenient recurrence relations for diagrams that enter the collinear-collinear contribution. Even the global recurrence relations for the unexpanded diagram turned out to be simpler, with the solutions described in Section 2. Although a systematical evaluation of the (c-c) contribution ot an arbitrary order in the expansion is certainly possible, it would be difficult to guess the analytic form ot the result by evaluating and studying first

terms of the expansion. Still we really used, in two points, the method of asymptotic expansions for crucial checks of our explicit procedure. Firstly, we have used the possibility to evaluate an arbitrary term of the (h-h) contribution (both in the limits ($t/s \to 0$ and $s/t \to 0$) for checking (global) recurrence relations and their solutions presented in Section 2. This was possible because any recurrence relation derived within integration by parts commutes with the (h-h) expansion which, by definition, is performed under the sign of the integrals involved (either in the integrals in the loop momenta or in the α-parameters). Secondly, we have evaluated the LO and NLO contributions of the first master double box and successfully compared them with the expansion of the analytical result.

In fact, we could avoid the rather cumbersome evaluation of the first master double box by starting from equations (29) and (30), applying solutions of recurrence relations of Section 2, evaluating all simpler diagrams, eliminating the second master double box by use of the second equation and arriving at a second order differential equation for the first master double box. It would be possible to solve this equation by expanding in t/s. But then we would need boundary conditions to solve it. Here we could insert the first two orders of the expansion in the limit $t/s \to 0$ evaluated by means of the strategy of regions as explained in Section 5 (where we have met only 3-fold rather 5-fold MB integrals.) However, in this case, we would not have crucial checks for the obtained results.

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